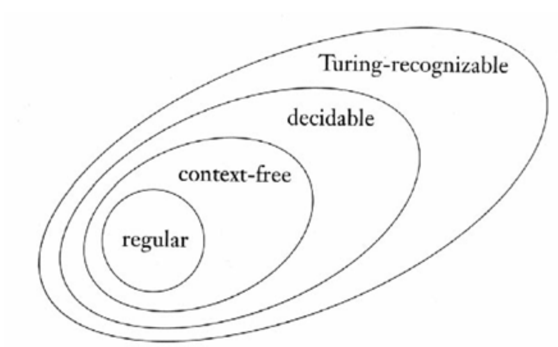
# Outline

- – Language Hierarchy
- – Definition of Turing Machine
- – TM Variants and Equivalence
- – Decidability
- – Reducibility

# Language Hierarchy




•Regular: finite memory

•CFG/PDA: infinite memory but in stack space

•TM: infinite and unrestricted memory

–TM Decidable/Recursive

–TM Recognizable/Recursively Enumerable

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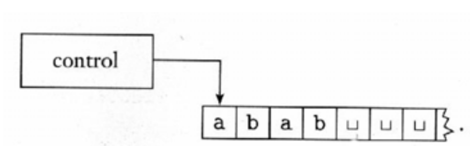
# Semantics of TM




Alan Turing (1912-1954)

- Not a real machine, but a model of computation
- Components:
  - 1-way infinite tape: unlimited memory
    - Store input, output, and intermediate results
    - Infinite cells
    - Each cell has a symbol from a finite alphabet
  - Tape head:
    - Point to one cell
    - Read or write a symbol to that cell
    - move left or right

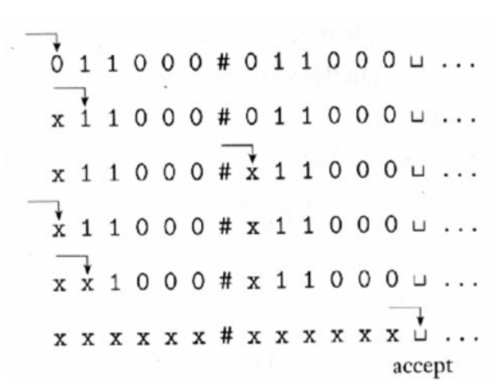
# States of a TM

- Initial state:
  - Head on leftmost cell
  - input on the tape
  - Blank  everywhere else
- Accept state
- Reject state
- Loop
- Accept or reject immediately

# An Example

$B = \{w\#w | w \in \{0,1\}^*\}$, and $B = L(M_1)$

- The tape changing:



# Formal Definition

A **Turing machine** is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where $Q$, $\Sigma$, and $\Gamma$ are all finite sets and

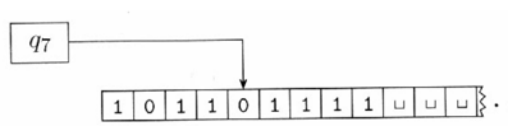
1. $Q$ is the set of states,
2. $\Sigma$ is the input alphabet, where the *blank* symbol $\sqcup \notin \Sigma$,
3. $\Gamma$ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. $\delta : Q \times \Gamma \longrightarrow Q \times \Gamma \times \{L, R\}$ is the transition function,
5. $q_0 \in Q$ is the start state,
6. $q_{\text{accept}} \in Q$ is the accept state, and
7. $q_{\text{reject}} \in Q$ is the reject state.

Example of transition function:

$$\delta(q, a) = (p, b, L)$$
$$\delta(q, a) = (p, b, R)$$

# Configuration

- A configuration of TM:
  - Current state
  - Symbols on tape
  - Head of location
- A formal specification of a configuration:
  - $uqv$, where
  - $u, v$ are strings on $\Gamma$, and $uv$ is the current content on taps
  - q is current state
  - head is in the first symbol of $v$.
  - ex: 1011 $q_7$ 01111



# Configuration

- For two configurations:
  $uaq_ibv$ and $uq_jacv$, where
  $a, b, c \in \Gamma$, and $u, v \in \Gamma^*$
  $uaq_ibv$ **yields** $uq_jacv$ if $\delta(q_i, b) = (q_j, c, L)$
  $uaq_ibv$ **yields** $uacq_jv$ if $\delta(q_i, b) = (q_j, c, R)$

- Two special cases:
  - the leftmost cell
    - $q_ibv$ yields $q_jcv$ for $\delta(q_i, b) = (q_j, c, L)$
    - $q_ibv$ yields $cq_jv$ for $\delta(q_i, b) = (q_j, c, R)$
  - on the cell with blank symbol
  - $uaq_i$ is equivalent to $uaq_i\sqcup$

# Configuration

- Initial configuration with input *w*: $\boldsymbol{q_0 w}$
- Accepting configuration: $\boldsymbol{uq_{accept}v}$
- *Rejecting configuration:* $\boldsymbol{uq_{reject}v}$
- $uq_{accept}v$ and $uq_{reject}v$ do not yield any other configurations
  - Immediate effect of accepting/rejecting
  - Halting configurations
- For a TM *M, a* string $w \in L(M)$ if there is a sequence of configurations $C_1, C_2, \ldots C_k$ such that:
  - $C_1 = q_0 w$
  - $C_i$ yields $C_{i+1}$ for $1 \le i \le k$
  - $C_k = uq_{accept}v$, $u, v \in \Gamma^*$

# Languages

- Turing-recognizable Languages:
  - For a $L \subseteq \Gamma^*$, exists a $M$ such that $M$ **recognizes** $L$
  - "Recognize" means accept, reject, or loop
- Turing-decidable languages:
  - For a $L \subseteq \Gamma^*$, exists a M such that $M$ **decides** $L$
  - "Decide" means halting: either accept or reject
- Turing-decidable $\subset$ Turing-recognizable
  - Halting Problem is Turing-recognizable, but not decidable.
- Not all languages are Turing-recognizable
  - There are some languages cannot be recognized by a TM.
    - Complement of Halting problem is Turing-unrecognizable

# An example

$A = L(M_2)$, where $A = \{0^{2^n} | n \geq 0\}$

- Semantical description:

*For an input string w:*

*{ sweep left to the right along the tape, crossing off every other 0*

***if** tape contains single 0*

*{ return accepted;}*

***elseif** tape contains odd number and more than one of 0s*

*{ return (rejected);}*

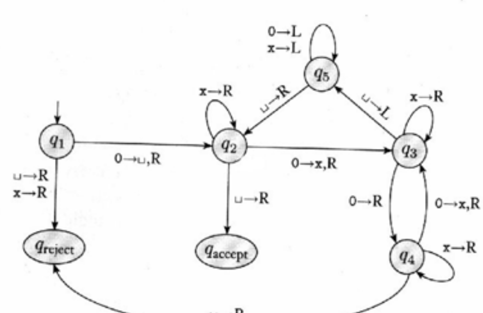
***else** go back to leftmost cell;*

*}*

- Formal description:

$M_2 = \{Q, \Sigma, \Gamma, \delta, q_1, q_{accept}, q_{reject}\}$, where

- $Q = \{q_1, q_2, q_3, q_4, q_5, q_{accept}, q_{reject}\}$
- $\Sigma = \{0\}$
- $\Gamma = \{0, x, \sqcup\}$
- $\delta$: state transition diagram



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# TM Variants

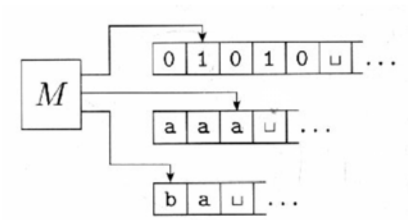
- Multitape TM
- Nondeterministic TM
- Enumerators
- Equivalence:All have same power
  – Recognize the same class of languages
  – Can be simulated by an ordinary TM

# Simple variant

- $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$
- $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, RR, LL\}$
- They are equivalent in recognizing language:
  - They can be simulated by original the TM
  - The difference is not significant

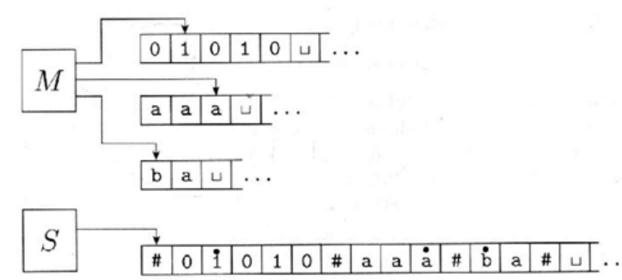
# Multitape TM

- A multitape TM is identical to ordinary TM except:
  - $k$ tapes, where $k \geq 1$
  - Each tap has its own head
  - $\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R, S\}^k$
  - $\delta(q_i, a_1, a_2, \ldots, a_k) = (q_j, b_1, b_2, \ldots, b_k, L, R, \ldots, R)$



# Multitape TM

- Theorem: each multitape TM has an equivalent single tape TM
  - Put # in a single tape for demarcation of original *k* tapes.
  - Each movement of *M* is simulated by a series movement of *S* on each segment.
  - For a right-move on the rightmost cell of *i*th tape in *M*, *S* write blank symbol in *(i+1)th* #, and right-shifts all symbols after that one cell.

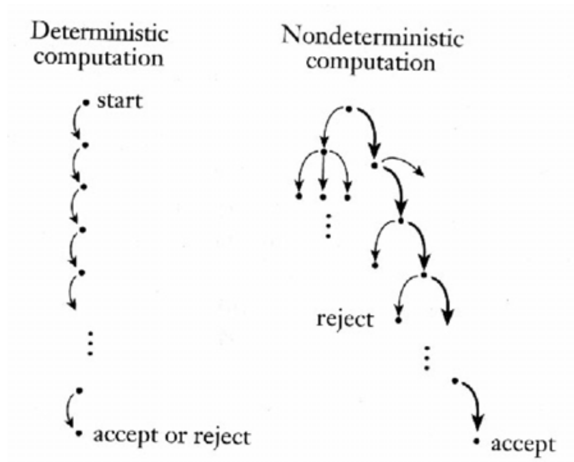



# Nondeterministic TM

- A nondeterministic TM is identical to an ordinary TM except:
  - $\delta : Q \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$
  - At any point the head has several possibilities to read/write/move.
- In deterministic TM, a computation is a single path with sequence of configurations.
- In nondeterministic TM, a computation is a tree or a directed acyclic graph.
  - A NTM accepts an input string if there exists a path leading to an accept state.
  - If all paths lead to reject state, then this input is rejected.

# NTM

- A computation single path and multi-path in a tree:

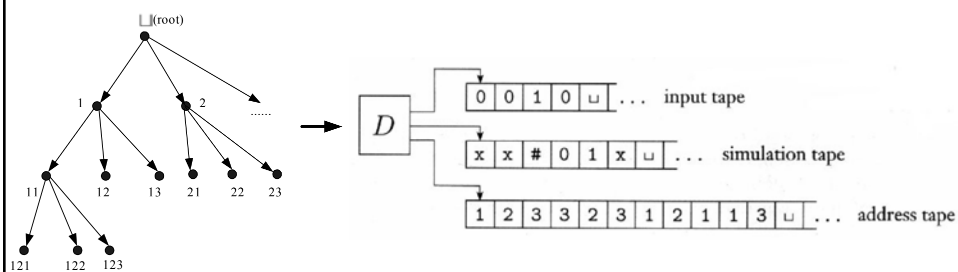



# Nodeterminism

- Is nondeterministic model always equivalent to a deterministic model?
  – Yes, for FA
  – No, for PDA
    • Some CFL cannot be recognized by any DPDA.
  – Yes, for TM!

# NTM

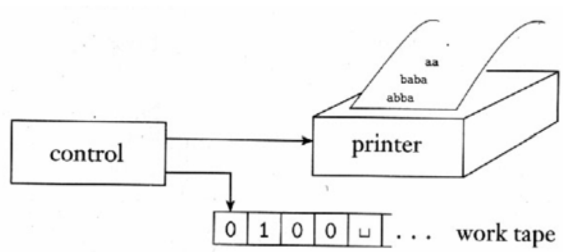
- Theorem: *Every NTM has an equivalent DTM.*




- For a computing tree of a NTM *N* with an input *w*, simulated with a 3-tape DTM *M*:
  - 1st tape: input *w*
  - 2nd tape: tape of a computing path with *N*
  - 3rd tape: node address (finite)

# Enumerator

- Semantically, an enumerator is a TM with an attached printer.
- Every time the TM wants to add a string to its output list, it sends the string to the printer.
- The language enumerated by an enumerator *E* is the collection of all the strings that *E* eventually prints out.



# Enumerator

- Theorem: *A language is Turing-recognizable iff some enumerator enumerates it.*
  - For a language, if $E$ enumerates it, then construct a TM $M$ works as:
    - Run $E$. Every time that $E$ outputs a string, compare it with input $w$.
    - If $w$ appears in the output of $E$, *accept*.
  - For a language recognized by a TM $M$, construct $E$ such that:
    - Run $M$ for $i$ steps on each input, $s1$, $s2$, …, $si$.
    - If any computations accept, print out the corresponding $sj$.
    - Repeat the above two steps with all possible inputs
- An enumerator can be regarded as a 2-tape TM.
  - Write accepted list on the 2nd tape.

# Other Variants

- Write-twice TM
  - Each cell on tape can only be written twice
- Write-once TM
  - Each cell on tape can only be written once
- TM with doubly infinite tape
  - Two-way infinite tape
- Universal TM
  - A TM that takes input of description of another TM.

# Thesis

- Church-Turing Thesis:
  - *Any algorithm can be expressed as a TM*
  - Formally defines an algorithm:

| | | |
|---|---|---|
| *Intuitive notion of algorithms* | equals | *Turing machine algorithms* |

- Extended Church-Turing Thesis:
  - *Any polynomial-time algorithm can be expressed as a TM that operates in polynomial time.*
  - A polynomial-time algorithm: number of element operations is a polynomial function of input length.
  - A polynomial-time TM: number of state transition is a polynomial function of input length.

# Describing TM

- Formal description
  - specifying Turing machine's states, transition function, and so on.
- Implementation description
  - using natural language to describe the way that the Turing machine moves its head and the way that it stores data on its tape.
- High-level description
  - using natural language describe an algorithm, ignoring the implementation model.

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# Solvability

- Solvable:
  – an algorithm to solve it,
  – a TM decides it.
- Unsolvable:
  – not algorithm to solve it
  – no TM can decide it.

# Decidable Language

$A_{\mathsf{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts } w\}$

- Acceptance problem:
  - Whether a particular DFA *B* accepts a given input string *w*.
- Membership problem:
  - Another way to say: whether <*B,w*> is a member of $A_{DFA}$.
- Theorem*: $A_{DFA}$ is a decidable language.*

$M =$ "On input $\langle B, w\rangle$, where $B$ is a DFA and $w$ is a string:

1. Simulate $B$ on input $w$.
2. If the simulation ends in an accept state, *accept*; otherwise, *reject*."

# Decidable Language

$A_{\mathsf{NFA}} = \{\langle B, w\rangle \mid B \text{ is an NFA that accepts } w\}.$

- Theorem*: $A_{NFA}$ is a decidable language.*

$N =$ "On input $\langle B, w\rangle$, where $B$ is an NFA and $w$ is a string:

1. Convert NFA $B$ to an equivalent DFA $C$.
2. Run TM $M$ for deciding $A_{\mathsf{DFA}}$ (as a "procedure") on input $\langle C, w\rangle$.
3. If $M$ accepts, *accept*; otherwise, *reject*."

# Decidable Language

$A_{\mathsf{REX}} = \{\langle R, w\rangle \mid R$ is a regular expression that generates $w\}$

- Theorem: *$A_{REX}$ is a decidable language.*

$P =$ "On input $\langle R, w\rangle$, where $R$ is a regular expression and $w$ is a string:

1. Convert regular expression $R$ to an equivalent DFA $A$.
2. Run TM $M$ for deciding $A_{\mathsf{DFA}}$ on input $\langle A, w\rangle$.
3. If $M$ accepts, *accept*; otherwise, *reject*."

# Decidable Language

$E_{\mathsf{DFA}} = \{\langle A\rangle \mid A$ is a DFA and $L(A) = \emptyset\}$

- Emptiness test problem:
  - Whether the language of a particular DFA is empty.
- Theorem: *$E_{DFA}$ is a decidable language.*

$T =$ "On input $\langle A\rangle$, where $A$ is a DFA:

1. Mark the start state of $A$.
2. Repeat Step 3 until no new states get marked.
3. Mark any state that has a transition coming into it from any state that is already marked.
4. If no accept state is marked, *accept*; otherwise, *reject*."

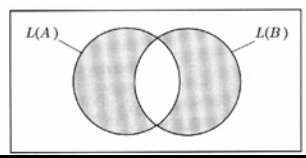
# Decidable Language

$EQ_{\mathsf{DFA}} = \{ \langle A, B \rangle \mid A$ and $B$ are DFAs and $L(A) = L(B) \}$

- Equivalence problem:
  - Test whether two DFAs recognize the same language.
- Theorem: *$EQ_{DFA}$ is a decidable language.*

$F =$ "On input $\langle A, B \rangle$, where $A$ and $B$ are DFAs:

1. Construct DFA $C = (A \cap \overline{B}) \cup (\overline{A} \cap B)$.
2. Run TM $T$ for deciding $E_{\mathsf{DFA}}$ on input $\langle C \rangle$.
3. If $T$ accepts, *accept*; otherwise, *reject*."




# Other Problems

- $A_{CFG}$ is decidable.
- $E_{CFG}$ is decidable.
- $EQ_{CFG}$ is undecidable.
  - CFG is not closed in intersection and complementation.

- $A_{TM}$ is undecidable.
  - Halting problem
- $E_{TM}$ is undecidable.
- $EQ_{TM}$ is undecidable.

# Halting Problem

$$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$$

Theorem: *$A_{TM}$ is Turing-recognizable*.

$U =$ "On input $\langle M, w\rangle$, where $M$ is a TM and $w$ is a string:

1. Simulate $M$ on input $w$.
2. If $M$ ever enters its accept state, *accept*; if $M$ ever enters its reject state, *reject*."

- $U$ is an example of universal TM.
- $U$ keeps looping if M neither accepts or rejects.

# Halting Problem

- Theorem: *$A_{TM}$ is undecidable.*
  - Can be proved by recursive theorem.

Suppose $H$ is a decider for $A_{\mathsf{TM}}$:

$$H(\langle M, w\rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ accepts } w \\ \textit{reject} & \text{if } M \text{ does not accept } w \end{cases}$$

$D =$ "On input $\langle M\rangle$, where $M$ is a TM:

1. Run $H$ on input $\langle M, \langle M\rangle\rangle$.
2. If $H$ accepts, *reject* and if $H$ rejects, *accept*."

$$D(\langle \mathsf{M}\rangle) = \begin{cases} \textit{accept} & \text{if } \mathsf{M} \text{ does not accept } \langle \mathsf{M}\rangle \\ \textit{reject} & \text{if } \mathsf{M} \text{ accepts } \langle \mathsf{M}\rangle \end{cases}$$

$$D(\langle D\rangle) = \begin{cases} \textit{accept} & \text{if } D \text{ does not accept } \langle D\rangle \\ \textit{reject} & \text{if } D \text{ accepts } \langle D\rangle \end{cases}$$

# Unrecognizable

- Theorem: *There are languages that cannot recognized by any TM.*
  - The set of TMs are countable
    - $Q$, $\Sigma$, and $\Gamma$ are all finite sets
    - Number of transition functions is countable.
  - The set of languages is uncountable.
    - $w \in \Gamma^*$
    - $L \subseteq \Gamma^*$
    - $L \in \mathcal{P}(\Gamma^*)$, $\mathcal{P}(\Gamma^*)$ is uncountable
      - Diagonalization method to prove this

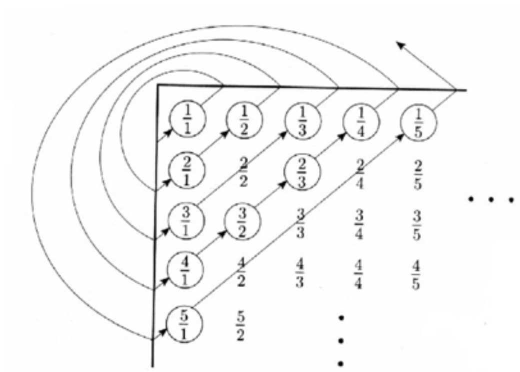
# Countable and Uncountable

- Two infinite sets *A* and *B* are the **same size** if there is a correspondence from A to B.
  - A correspondence is a one-to-one and onto function: $f : A \rightarrow B$
  - one-to-one: $f(a) \neq f(b)$ whenever $a \neq b$
  - Onto: $\forall b \in B, \exists a \in A, f(a) = b$

- A set is **countable** if either it is finite or it has the same size as $N$ ={1,2,3…}; otherwise it is **uncountable**.

# Countable

- Set of position rational numbers is countable: $\{m/n, m, n \in \mathcal{N}\}$




# Uncountable

- Set of real numbers *R* is uncountable:

Assume that a correspondence $f$ existed between $\mathcal{N}$ and $\mathcal{R}$.

| $n$ | $f(n)$ |
|---|---|
| 1 | $3.\underline{1}4159\cdots$ |
| 2 | $55.5\underline{5}555\cdots$ |
| 3 | $0.12\underline{3}45\cdots$ |
| 4 | $0.500\underline{0}0\cdots$ |
| $\vdots$ | $\vdots$ |

We can find an $x$, $0 < x < 1$, so that the $i$-th digit following the decimal point of $x$ is different from that of $f(i)$; for example, $x = 0.4641\cdots$ is a possible choice.

## Uncountable

- The set of all languages over an alphabet is uncountable.
  - Think that a real number is a string over alphabet of {. , 0,1,2,3,4,5,6,7,8,9}
  - Similar diagonalization way to prove with general alphabet

- Theorem: *__A language is decidable iff both it and its complement language are Turing-recognizable.__*
  - If $A$ is decided by $M_1$, then :
    - $M_2$="on input $w$:
      1. Run $M_1$ on $w$.
      2. If $M_1$ rejects, *accept*; if $M_1$ accepts, *reject*. "
    - $M_2$ decides $\overline{A}$
  - If A and $\overline{A}$ are Turing-recognizable:

    Let $M_1$ be a recognizer for $A$ and $M_2$ be a recognizer for $\overline{A}$.

    $M =$ "On input $w$:

    1. Run both $M_1$ and $M_2$ on input $w$ in parallel. ($M$ takes turns simulating one step of each machine until one of them halts.)
    2. If $M_1$ accepts, *accept* and if $M_2$ accepts, *reject*."

$\overline{A_{\mathsf{TM}}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ does not accept } w\}$

- Theorem: $\overline{A_{\mathsf{TM}}}$ *is not Turing-recognizable*
  - If $\overline{A_{\mathsf{TM}}}$ is Turing-recognizable, and $A_{TM}$ is Turing-recognizable, then $A_{TM}$ must be decidable.—contradiction!

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# Reducibility

- Semantics
- Reduce $A_{TM}$ to $HALT_{TM}$
- PCP Problem
- Mapping Reducibility